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INSPHERES AND INNER PRODUCTS

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ABSTRACT
Among normed linear spaces X of dimension $\geqq 3$, finite-dimensional Hilbert spaces are characterized by the condition that for each convex body C in X and each ball B of maximum radius contained in C, B's center is a convex combination of points of $B \cap$ (boundary of C). Among reflexive Banach spaces of dimension $\geqq 3$, general Hilbert spaces are characterized by a related but weaker condition on inscribed balls.

Introduction

Throughout this paper, X denotes a real normed linear space. For a subset Y of X, co Y, cl Y, int Y and ∂Y denote respectively the convex hull, the closure, the interior, and the boundary of Y in X. For each $\rho > 0$,

$$S_\rho = \{x \in X : \|x\| = \rho\} \quad \text{and} \quad B_\rho = \{x \in X : \|x\| \leqq \rho\}.$$

A *sphere* ⟨resp. *ball*⟩ in X is a set of the form $S = c + S_\rho$ ⟨$B = c + B_\rho$⟩ for a point $c \in X$ and a number $\rho > 0$; these are the *center* and *radius.* A sphere S is an *insphere* ⟨resp. *circumsphere*⟩ of a set $W \subset X$ if the ball co S is contained in ⟨contains⟩ W but this is not true for any ball of larger ⟨smaller⟩ radius. A *convex body* is a set that is bounded, closed, convex and has nonempty interior.

If the normed linear space X is incomplete and B is a ball in the completion $\tilde{X}$ that is centered at a point of $\tilde{X} \sim X$, then the set $B \cap X$ is a convex body in X that has neither insphere nor circumsphere. If X is reflexive then each bounded subset of X admits a circumsphere (an easy consequence of the weak lower

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semicontinuity of the norm) and each convex body admits an insphere (see [13], where it is also shown that convexity is necessary in this claim). However, circumspheres and inspheres may behave differently (with respect to convex bodies) in nonreflexive Banach spaces. In fact, for each bounded set in a Banach space X to admit a circumsphere, it is sufficient for X to be a dual space or, more general, for X to be constrained in X^{**} (i.e., there exists a projection of norm 1 from X^{**} onto X) ([4]). On the other hand, Garkavi [6] has characterized both reflexivity and finite-dimensionality of X in terms of inspheres:

X is reflexive if and only if each convex body in X admits an insphere;

X is finite-dimensional if and only if for each convex body C in X and each insphere S of C, $S \cap \partial C \neq \emptyset$.

The main purpose of the present paper is to characterize Hilbert spaces (i.e., complete inner product spaces) in terms of the behavior of inspheres:

If $\dim X \geqq 3$ *then X is a finite-dimensional Hilbert space if and only if for each convex body C in X and each insphere $c + S_\rho$ of C, $c \in \mathrm{co}((c + S_\rho) \cap \partial C)$*;

if X is reflexive and $\dim X \geqq 3$ *then X is a Hilbert space if and only if for each convex body C in X and each insphere $c + S_\rho$ of C, $c \in \mathrm{cl}\,\mathrm{co}((c + B_\tau) \cap \partial C)$ for all $\tau > \rho$.*

These and some slightly sharper characterizations arise from a conjecture of Franchetti and Papini [3], and we are indebted to H. Berens for calling our attention to this conjecture.

The characterizations of Hilbert spaces are established in Section 2 after developing one of our main tools, the "canopy" construction, in Section 1. The other main tool is the Blaschke–Kakutani theorem [2] [8] characterizing Hilbert spaces in terms of projections of norm 1. It has also been used [5] [9] to characterize Hilbert spaces in terms of circumspheres.

Section 3 contains some additional constructions related to inspheres, and characterizes finite-dimensionality in terms of circumspheres:

X is finite-dimensional if and only if for each convex body C in X and each circumsphere S of C, $S \cap C \neq \emptyset$.

1. Hoods and canopies

In the normed linear space X, let Q be a closed halfspace whose bounding hyperplane ∂Q passes through the origin. For each $\delta \geqq 0$, we define the *hood*

$$H(Q,\delta) = \mathrm{co}[(B_1 \cap Q) \cup (B_{1+\delta} \cap \partial Q)]$$

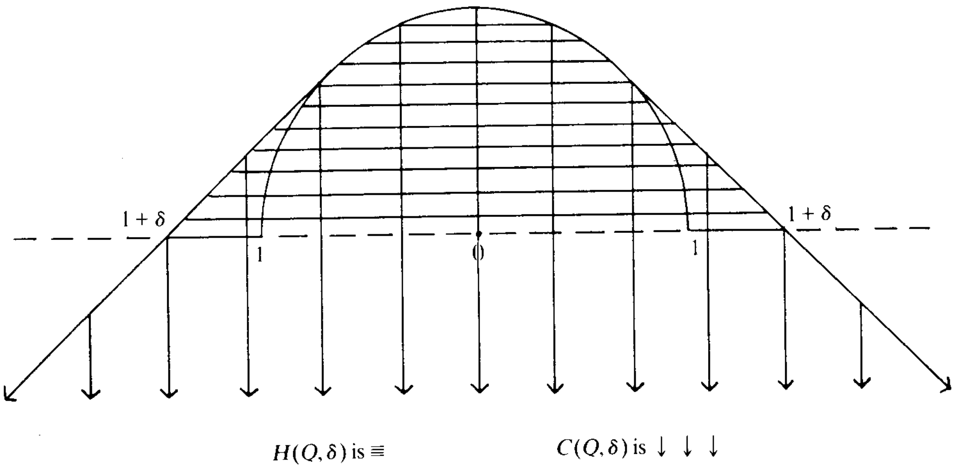

Fig. 1.

and the *canopy* $C(Q,\delta)$, which is the intersection of all closed halfspaces J in X such that $J \supset H(Q,\delta)$ and the hyperplane ∂J includes a point of $S_1 \cap (\operatorname{int} Q)$. (See Fig. 1.) When X is reflexive but not a Hilbert space, the canopy construction will be used in the next section to produce a convex body in X whose inspheres behave badly.

Our use of canopies is based on Proposition 1.1 and Theorem 1.5 below; the results 1.2–1.4 are lemmas of a technical nature.

1.1. Proposition. *For each $\delta > 0$ the intersection $S_1 \cap \partial C(Q,\delta)$ is interior to Q and at positive distance from ∂Q. Hence*

$$0 \notin \operatorname{cl}\operatorname{co}(S_1 \cap \partial C(Q,\delta)).$$

Proof. That $S_1 \cap \partial C(Q,\delta) \subset \operatorname{int} Q$ follows from the fact that if $x \in \operatorname{cl}(X \sim Q)$ and $x \notin C(Q,\delta)$ then $\|x\| > 1 + \delta$. To establish this fact, note that by the definition of $C(Q,\delta)$ there is a point $z \in S_1 \cap \operatorname{int} Q$ and a halfspace $J \supset H(Q,\delta)$ such that $z \in \partial J$ and $x \notin J$. The segment $[x, z[$ lies in $X \sim J$, and if y is the point at which this segment intersects ∂Q then $\|y\| > 1 + \delta$, which with $\|z\| = 1$ implies $\|x\| > 1 + \delta$.

To complete the proof of 1.1 we show that if $p \in S_1 \cap \partial C(Q,\delta)$ and $q \in \partial Q$, then $\|p - q\| > \delta/4$ if $0 < \delta < 1$ (whence of course $\|p - q\| \geqq \frac{1}{4}$ if $\delta \geqq 1$). This is obvious when $q = 0$, so suppose $q \neq 0$ and $\|p - q\| \leqq \delta/4$, and let $r = q/\|q\|$. Since $\|p\| = 1$, $1 - \delta/4 \leqq \|q\| \leqq 1 + \delta/4$, whence $\|q - r\| \leqq \delta/4$ and $\|p - r\| \leqq \delta/2$. Since $p \in \partial C(Q,\delta)$, there exists $f \in X^*$ such that $\|f\| = 1$ and the maximum of f on $C(Q,\delta)$ is attained at p. But $S_1 \subset C(Q,\delta)$ and $p \in S_1$ so this maximum is 1, whence $f[(1+\delta)r] \leqq 1$ because $(1+\delta)r \subset H(Q,\delta) \subset C(Q,\delta)$. We now reach the contradictory conclusion that

$$\delta = (1+\delta) - 1 \leqq f[(1+\delta)p] - f[(1+\delta)r] \leqq \|f\|(1+\delta)\|p-r\| \leqq (1+\delta)\delta/2 < \delta.$$

■

1.2. LEMMA. $\bigcap_{\delta>0} H(Q,\delta) = H(Q,0) = B_1 \cap Q = C(Q,0) \cap Q.$

PROOF. It is obvious that

$$Q \supset \bigcap_{\delta>0} H(Q,\delta) \supset H(Q,0) = B_1 \cap Q \subset C(Q,0) \cap Q.$$

Now choose $q \in \operatorname{int}(B_1 \cap Q)$ and consider an arbitrary $p \in Q \sim B_1$. The segment $]p,q[$ intersects S_1 at a point $v \in \operatorname{int} Q$, and there is a closed halfspace $J \supset S_1$ such that $v \in \partial J$. Since $p \notin J$ we conclude that $p \notin C(Q,0)$ and hence $C(Q,0) \cap Q = B_1 \cap Q$. Further, J admits a translate that misses p but contains $B_{1+\delta}$ for some $\delta > 0$, whence $p \notin H(Q,\delta)$ and it follows that $\bigcap_{\delta>0} H(Q,\delta) = B_1 \cap Q$. ■

1.3. LEMMA. *If the space X is reflexive then*

$$C(Q,\delta) \cap Q = H(Q,\delta) \qquad \textit{for each } \delta \geqq 0,$$

and

$$\bigcap_{\delta>0} C(Q,\delta) = C(Q,0).$$

PROOF. To establish the first equality we show that $p \in Q \sim H(Q,\delta)$ implies $p \notin C(Q,\delta)$, i.e., there is a closed halfspace $K \supset H(Q,\delta)$ such that $p \notin K$ and ∂K supports $H(Q,\delta)$ at a point of $S_1 \cap (\operatorname{int} Q)$. First notice that if $p \in \partial Q \sim H(Q,\delta)$ and $p \in C(Q,\delta)$, then for any point $x \in H(Q,\delta) \cap (\operatorname{int} Q)$ the open segment $]p,x[$ lies in $C(Q,\delta) \cap (\operatorname{int} Q)$, and since $\|p\| > 1+\delta$ the segment contains points not in $H(Q,\delta)$. Thus it suffices to consider $p \in (\operatorname{int} Q) \sim H(Q,\delta)$, and here it is obvious that

$$\operatorname{int} Q \supset]p,0[\cap \partial H(Q,\delta) \neq \emptyset.$$

With $0 < \lambda < 1$ and

$$v = \lambda p \in]p,0[\cap \partial H(Q,\delta),$$

let K be a closed halfspace that contains 0 and whose bounding hyperplane ∂K supports $H(Q,\delta)$ at v. Plainly $p \notin K$, and if $v \in S_1$ the argument is complete. If $v \notin S_1$ note that since X is reflexive the convex sets $B_1 \cap Q$ and $B_{1+\delta} \cap \partial Q$ are weakly compact and hence $H(Q,\delta)$, the convex hull of their union, is closed. Thus the point v of $\operatorname{int} Q$ belongs to an open segment $]s,t[$ for some $s \in B_1 \cap (\operatorname{int} Q)$ and $t \in B_{1+\delta} \cap \partial Q$. Since ∂K supports $H(Q,\delta)$ at v, and s and t both belong to $H(Q,\delta)$, it follows that $s \in S_1 \cap (\operatorname{int} Q)$ and ∂K supports $H(Q,\delta)$ at s. We conclude that $C(Q,\delta) \cap Q = H(Q,\delta)$.

For the second part of the proof, consider an arbitrary point $p \in X \sim C(Q,0)$, and let the closed halfspace $J \supset B_1 \cap Q$ and point $v \in S_1 \cap (\operatorname{int} Q)$ be such that $p \notin J$ and ∂J supports $B_1 \cap Q$ at v. Let $w \in]p,v[\cap Q$, whence $w \in Q \sim H(Q,0)$ and from Lemma 1.2 and the equality of the first part it follows that $w \notin C(Q,\delta)$ for some $\delta > 0$. There is a closed halfspace $K \supset H(Q,\delta)$ such that $w \notin K$ and ∂K supports $H(Q,\delta)$ at a point of $S_1 \cap (\operatorname{int} Q)$. Since $v \in K$ but $w \notin K$, the point p is not in K and hence not in $C(Q,\delta)$. ■

1.4. LEMMA. *Suppose the sequences $t_1, t_2, \ldots, \mu_1, \mu_2, \ldots$ and $\delta_1, \delta_2, \ldots$ are such that the t_i's are points of a reflexive space X, the μ_i's are positive numbers such that the sequence $\mu_1 t_1, \mu_2 t_2, \ldots$ converges weakly to a point $t \in X \sim \{0\}$, and the δ_i's are positive numbers converging to 0 such that*

$$(S_1 \cap Q) + t_i \subset C(Q,\delta_i) \qquad \text{for each } i.$$

Then

$$(S_1 \cap Q) + t \subset C(Q,0) \quad \text{and} \quad (B_1 \cap \partial Q) + \mathbf{R}t = B_1 + \mathbf{R}t,$$

where $\mathbf{R}t$ is the line through the origin consisting of all multiples of t.

PROOF. Suppose first that $(S_1 \cap Q) + t \not\subset C(Q,0)$, whence by Lemma 1.3 there exist $v \in S_1 \cap Q$ and $\delta > 0$ such that $v + t \notin C(Q,\delta)$. Hence there is a point $p \in S_1 \cap (\operatorname{int} Q)$ and a closed halfspace $J \supset H(Q,\delta)$ such that $p \in \partial J$ and $v + t \notin J$. If $f \in X^*$ and $J = \{x : f(x) \leqq 1\}$ then $f(p) = 1$, $f(v) \leqq 1$ and $f(t+v) > 1$, whence

$$f(p+t) = f(v+t) + f(p) - f(v) > 1.$$

Since $p + t_i \in C(Q,\delta)$ we have

$$0 \geqq f(p+t_i) - f(p) = [f(p+\mu_i t_i) - f(p)]/\mu_i.$$

But $p + \mu_i t_i$ converges weakly to $p + t$ and $f(p+t) > 1$, so eventually the right side of the last equation is positive and a contradiction ensues. Thus we have proved

$$(S_1 \cap Q) + t \subset C(Q,0).$$

By changing the multipliers μ_i, the point t may be replaced by an arbitrary positive multiple of itself, and consequently

$$(\dagger) \qquad S_1 \cap Q + [0,\infty[t \subset C(Q,0).$$

It follows from this that $t \notin Q$, for $t \neq 0$ and $C(Q,0) \cap Q$ is (by Lemma 1.1) the bounded set $B_1 \cap Q$.

Up to this point, we have used the fact that $0 \in \operatorname{int} B_1$ but have not in any way used the fact that B_1 is symmetric about 0. Now, however, that fact is needed. Plainly

$$B_1 + \mathbf{R}t \supset (B_1 \cap \partial Q) + \mathbf{R}t.$$

If the reverse inclusion fails there exist $v \in S_1$ and $\lambda \neq 0$ such that the point $v + \lambda t$ does not belong to $(B_1 \cap \partial Q) + \mathbf{R}t$ and consequently the entire line $v + \mathbf{R}t$ misses the intersection $B_1 \cap \partial Q$. By symmetry the same is true of $-v + \mathbf{R}t (= -v - \mathbf{R}t)$ and hence we may assume without loss of generality that $v \in Q$. With $t \notin Q$, there exists $\alpha \geqq 0$ such that $v + \alpha t \in \partial Q$, whence by (†) and 1.1,

$$v + \alpha t \in C(Q,0) \cap \partial Q = B_1 \cap \partial Q,$$

a contradiction completing the proof. ■

1.5. Theorem. *Suppose that X is a reflexive space with unit sphere S_1, that Q is a closed halfspace in X whose bounding hyperplane ∂Q passes through the origin, and that X does not admit a projection of norm 1 onto ∂Q. Then there exists $\delta > 0$ such that S_1 is the unique insphere of the canopy $C(Q,\delta)$. The canopy is finitely bounded, and is norm-bounded when X is finite-dimensional.*

Proof. If S_1 is the unique insphere of the canopy $C(Q,\delta)$, then surely $C(Q,\delta)$ is finitely bounded (has bounded intersection with each finite-dimensional subspace of X). For otherwise, since $C(Q,\delta)$ is closed and convex, it fails to be linearly bounded and hence, with $0 \in C(Q,\delta)$, contains an entire ray $[0,\infty[p$ issuing from the origin. But then $\lambda p + S_1 \subset C(Q,\delta)$ for each $\lambda \geqq 0$, contradicting the uniqueness assertion on the insphere.

To prove that S_1 is the unique insphere, we proceed by contradiction. If the conclusion fails then for each i there is a sphere $c_i + S_{\rho_i} \subset C(Q,1/i)$ with either $\rho_i > 1$ or $\rho_i = 1$ and $c_i \neq 0$, and there is a sequence $t_1, t_2, \ldots$ in $X \sim \{0\}$ such that $t_i + S_1 \subset C(Q,1/i)$ for each i. Let $u_i = t_i/\|t_i\|$. Since the unit ball B_1 is sequentially compact we may assume (passing to a subsequence if necessary) the sequence $u_1, u_2, \ldots$ converges weakly to a point $t \in B_1$. If $t \neq 0$ and if P denotes the projection of X onto ∂Q whose kernel is the line $\mathbf{R}t$, then Lemma 1.4's equation

$((B_1 \cap \partial Q) + \mathbf{R}t = B_1 + \mathbf{R}t)$ amounts to saying that $PB_1 = B_1 \cap \partial Q$ and hence $\|P\| = 1$. This contradicts the assumption on ∂Q and hence completes the proof if we can show $t \neq 0$.

Suppose that $t = 0$ and let $g \in S_1(X^*)$ be such that $Q = \{x \in X : g(x) \geqq 0\}$. Then as $i \to \infty$, $\operatorname{dist}(u_i, \partial Q) = |g(u_i)| \to 0$, whence $\operatorname{dist}(u_i, S_1 \cap \partial Q) \to 0$ and

$$\operatorname{dist}(u_i, S_{3/2} \cap \partial Q) \to \tfrac{1}{2}.$$

Let the point $w \in S_{3/2} \cap Q$ and the integer $k > 2$ be such that $\|u_k - w\| < \frac{3}{4}$, and let z be a point of $X \sim C(Q, \frac{1}{2})$ such that $\|z - w\| < \frac{1}{4}$ (such a z must exist because Lemma 1.3 implies $w \in \partial C(Q, \frac{1}{2})$). Then a closed halfspace J must exist such that $J \supset C(Q, \frac{1}{2})$, ∂J supports $C(Q, \frac{1}{2})$ at a point of $S_1 \cap (\operatorname{int} Q)$, and $z \notin J$. Obviously

$$\operatorname{dist}(w, \partial J) \leqq \|z - w\| < \tfrac{1}{4}.$$

With $f \in S_1(X^*)$ such that $J = \{x \in X : f(x) \leqq 1\}$, this inequality implies

$$f(w) > \tfrac{3}{4}.$$

On the other hand, since $C(Q, 1/k) \subset C(Q, 1/2)$ we have

$$1 - f(t_k) = \operatorname{dist}(t_k, \partial J) \geqq \operatorname{dist}(t_k, \partial C(Q, \tfrac{1}{2})) \geqq \operatorname{dist}(t_k, \partial C(Q, 1/k)) \geqq 1,$$

whence $f(t_k) \leqq 0$ and $f(u_k) \leqq 0$. This leads to the contradictory conclusion that

$$\tfrac{3}{4} < f(w) - f(u_k) \leqq \|f\| \, \|w - u_k\| < \tfrac{3}{4}. \qquad \blacksquare$$

2. Hilbert spaces characterized by nice behavior of inspheres

The theorems below show that, as was suggested by Franchetti and Papini [3, p. 84], Hilbert spaces are characterized except in the two-dimensional case by the nice behavior of inspheres. The exception is familiar from the Blaschke–Kakutani characterization.

2.1. THEOREM. *The following four conditions are equivalent for each normed linear space X of dimension $\geqq 3$ with unit sphere S_1 and unit ball B_1:*

(i) *X is a finite-dimensional Hilbert space;*

(ii) $\dim X < \infty$ *and for each set W in X of which S_1 is an insphere,*

$$0 \in \operatorname{int} \operatorname{co}(B_\rho \sim W) \qquad \textit{for all } \rho > 1;$$

(iii) *for each set W in X of which S_1 is an insphere,* $0 \in \operatorname{co}(S_1 \cap \partial W)$;

(iv) *for each convex body C in X of which S_1 is the unique insphere,* $0 \in \operatorname{cl} \operatorname{co}(S_1 \cap \partial C)$.

This theorem will be proved along with the following.

2.2. Theorem. *The following four conditions are equivalent for each reflexive space X of dimension $\geqq 3$ with unit sphere S_1 and unit ball B_1:*

(i) *X is a Hilbert space;*

(ii) *for each closed set W in X of which S_1 is an insphere,*

$$0 \in \operatorname{int} \operatorname{co}(B_\rho \sim W) \qquad \text{for all } \rho > 1;$$

(iii) *for each set W in X of which S_1 is an insphere,*

$$0 \in \operatorname{cl} \operatorname{co}(B_\rho \cap \partial W) \qquad \text{for all } \rho > 1;$$

(iv) *for each convex body C in X of which S_1 is the unique insphere,*

$$0 \in \operatorname{cl} \operatorname{co}(B_\rho \cap \partial C) \qquad \text{for all } \rho > 1.$$

Proof of 2.1 and 2.2. Plainly (iii) implies (iv) in both cases, and canopies are used at the end of the argument to show (iv) implies (i) in both cases. The rest of the proof consists of showing that (i) implies (ii) in both cases, (ii) implies (iii) in 2.1, (ii) implies (iii) for closed W in 2.2 (so (iv) is implied), and (i) implies (iii) for arbitrary W in 2.2.

We will use the following simple fact:

(F) If Q is a closed halfspace in a Hilbert space, and Q's bounding hyperplane ∂Q contains the origin, then for each $\rho > 1$ the set $(B_1 \cap Q) \cup (B_\rho \sim Q)$ contains a ball of radius $(1+\rho^2)/(2\rho)$.

(i) $\Rightarrow$ (ii) Let $Y = B_\rho \sim W$ and note that in the case of 2.2 it follows from the closedness of W that int $Y \neq \emptyset$. (This fact is not needed in 2.1.) If (ii) does not hold then by standard separation theorems there is a closed halfspace $Q \supset Y$ with $0 \in \partial Q$. From $Y \subset Q$ it follows that $B_\rho \sim Q \subset W$ and hence, since $B_1 \subset W$, that

$$(B_1 \cap Q) \cup (B_\rho \sim Q) \subset W.$$

It then follows with the aid of (F) that S_1 is not an insphere of W.

(ii) $\Rightarrow$ (iii) in 2.1. Assume that $\dim X = d$ and S_1 is an insphere of W. For each positive integer n it follows from condition (ii) and Carathéodory's theorem that there are points

$$y_0(n), \ldots, y_d(n) \in B_{1+1/n} \sim W$$

such that $0 \in \operatorname{co}\{y_0(n), \ldots, y_d(n)\}$. Since B_2 is compact we may assume without loss of generality that for each $i \in \{0, \ldots, d\}$, the sequence $y_i(1), y_i(2), \ldots$ converges to a point z_i of B_2. It is clear that $0 \in \operatorname{co}\{z_0, \ldots, z_d\}$ and that each z_i belongs to S_1, hence to W, and thus, as the limit of a sequence in $X \sim W$, to ∂W.

(i) $\Rightarrow$ (iii) in 2.2, and (ii) $\Rightarrow$ (iii) for closed W in 2.2. Suppose there exists $\rho > 1$ such that $0 \notin \operatorname{cl}\operatorname{co}(B_\rho \cap \partial W)$. Then there is a closed halfspace Q such that $0 \in \partial Q$ and the set $B_\rho \cap \partial W$ is contained in a translate P of Q that is interior to Q. But then ∂W misses the set $B_\rho \sim P$, which is connected and intersects B_1. With $B_1 \subset W$, it follows that $B_\rho \sim P \subset W$. This contradicts the conclusion of 2.2 (ii) when W is closed, and it contradicts 2.2 (i) for an arbitrary W since it implies

$$(B_1 \cap Q) \cup (B_\rho \sim Q) \subset W$$

and then (F) is applicable.

(iv) $\Rightarrow$ (i). If (i) fails, there is a closed halfspace Q in X such that $0 \in \partial Q$ and X does not admit a projection of norm 1 onto ∂Q. By Theorem 1.5, there exists $\delta > 0$ such that S_1 is the unique insphere of the canopy $C(Q, \delta)$ and, by Proposition 1.1, $0 \notin \operatorname{cl}\operatorname{co}(S_1 \cap C(Q, \delta))$.

Under the hypothesis 2.1(iv), X is finite-dimensional, for Garkavi [5] shows that each infinite-dimensional normed linear space contains a centrally symmetric convex body C such that S_1 is the unique insphere of C and yet S_1 misses ∂C. Since $\dim X < \infty$, the set $C(Q, \delta)$ is bounded (by 1.5) and 2.1(iv) is contradicted.

Under the hypothesis 2.2(iv), consider the convex body

$$K = C(Q, \delta) \cap B_2$$

which has S_1 as its unique insphere. We claim that

$$0 \notin \operatorname{cl}\operatorname{co}(B_{1+\delta/3} \cap \partial K)$$

so that 2.2(iv) is contradicted. If our claim is false, the set

$$B_{1+\delta/3} \cap \partial K = B_{1+\delta/3} \cap \partial C(Q, \delta)$$

contains a sequence $x_1, x_2, \ldots$ of points such that as $i \to \infty$, $\operatorname{dist}(x_i, \partial Q) \to 0$. For each i, the argument used in the first part of the proof of Proposition 1.1 yields $x_i \in \operatorname{int} Q$, so $x_i \in \partial H(Q, \delta)$ by Lemma 1.3. Hence, in view of Proposition 1.1, $x_i \in S_1$ for each i, whence there are points $w_i \in S_{1+\delta} \cap \partial Q$ and $y_i \in S_1 \cap \partial K$ such that $x_i \in \,]w_i, y_i[$. Since

$$\sup_i \| w_i - y_i \| < \infty \quad \text{and} \quad \liminf_i \| w_i - x_i \| > \delta/3,$$

this implies $\operatorname{dist}(y_i, \partial Q) \to 0$, contradicting Proposition 1.1. ∎

3. Additional remarks, results and constructions

The various portions of this section (separated by *****) are unrelated to each other, but all are concerned with inspheres or circumspheres. We note first that in condition (ii) of 2.2, the requirement that W is closed may be replaced without change of argument by the requirement that W is convex, but not by the requirement that W is the union of two convex sets. For let X be an arbitrary infinite-dimensional normed linear space, let f be a discontinuous linear functional on X, and for each real λ let

$$W_\lambda = B_1 \cup \{x \in X : f(x) \geqq \lambda\}.$$

Then S_1 is the unique insphere of W_λ, but for each $\rho > 1$ the origin is not interior to the set $\mathrm{co}(B_\rho \sim W_\lambda)$, and when $\lambda < 0$ the origin does not even belong to this set.

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The property of reflexivity is characterized in terms of inspheres by the second result of Garkavi [6], and this combines with Theorem 2.2 to characterize Hilbert spaces in terms of inspheres, assuming only $\dim X \geqq 3$. However, we do not know whether, without assuming reflexivity, conditions (ii)–(iv) imply condition (i) in 2.2. It might be possible to approach this question with the aid of the theorem of Amir and Franchetti [1] (called to our attention by P. L. Papini) that characterizes Hilbert spaces for ($\dim X \geqq 3$) as those which admit projections of norm arbitrarily close to 1 onto each closed hyperplane through the origin.

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In this portion we assume X is an infinite-dimensional Banach space. As was mentioned earlier, Garkavi [6] shows that X contains a convex body C such that $C = -C$ and S_1 is the unique insphere of C, but $S \cap \partial C = \emptyset$. He gives two contructions, one for reflexive and one for nonreflexive spaces. Here we describe a single construction which assumes only that X has a separable infinite-dimensional closed linear subspace L with a quasicomplement M; that is, M is a closed linear subspace such that $L \cap M = \{0\}$ and $L + M$ is dense in X. It is known that such a pair (L, M) exists for any closed separable L if X itself is separable (Murray [12], Mackey [11]) or reflexive (Lindenstrauss [10]) and in many other cases (Rosenthal [14]), and it is believed that such a pair always exists. This is equivalent [14, p. 188] to requiring that there is a bounded linear transformation from X onto a separable infinite-dimensional space.

Let $x_1, x_2, \ldots$ be a sequence that is dense in $S_1 \cap L$, let $x_{-i} = -x_i$ for $i = 1, 2, \ldots$, and let

$$K = \operatorname{cl}\operatorname{co}\{2^{-|i|}x_i : i = \pm 1, \pm 2, \ldots\}.$$

Then K is compact and convex, and the subspace $\mathbf{R}K$ is dense in L. Let $C = \operatorname{cl}(B_1 + K + B_1 \cap M)$, a convex body with $C = -C$. We claim that S_1 misses ∂C and that $S_\rho \not\subset C$ for each $\rho > 1$.

Suppose there exists a point $p \in S_1 \cap (\partial C)$, whence p lies on a hyperplane that supports both B_1 and C and thus there exists $f \in X^*$ such that

$$1 = \|f\| = \|p\| = f(p) = \sup fC.$$

In particular, for each $k \in K$ and $q \in B_1 \cap M$ it is true that

$$f(p + k + q) \leqq f(p),$$

whence $f(k + q) \leqq 0$. Since K and $B_1 \cap M$ are both symmetric about 0, this implies that fR is identically 0 on K, hence on $\mathbf{R}K$ and (by density) on L; also, f is identically 0 on $B_1 \cap M$ and hence on M. But then f is identically 0 on $L + M$ and hence by density on X. This contradiction implies that S_1 misses ∂C.

Now note that for $n = 1, 2, \ldots$ there exists $f_n \in X^*$ with $\|f_n\| = 1$ and

$$f(M + \mathbf{R}x_1 + \cdots + \mathbf{R}x_n) = \{0\}.$$

If $S_\rho \subset C$ for some $\rho > 1$ then for each n there exists $y_n \in C$ such that $f_n(y_n) > (1 + \rho)/2$ and hence there exist $b_n \in B_1$, $k_n \in K$ and $m_n \in B_1 \cap M$ such that

$$f_n(b_n + k_n + m_n) > (1 + \rho)/2.$$

However, $f_n(b_n) \leqq 1$, $f_n(m_n) = 0$, and

$$f_n(k_n) \geqq 2 \sum_{i=n+1}^{\infty} 2^{-|i|} < (\rho - 1)/2$$

for all sufficiently large n, and the contradiction completes the proof that $S_\rho \not\subset C$ for each $\rho > 1$.

* * * * *

As was mentioned earlier, circumspheres have been used to characterize Hilbert spaces [5] [9]. We close by using them to characterize finite-dimensionality.

3.1. THEOREM. *A normed linear space X is finite-dimensional if and only if for each convex body C in X and each circumsphere S of C, S intersects C.*

PROOF. The "only if" part is obvious. For the "if" part, in the case of reflexive X the claim was proved by Garkavi [6, th. 5] (there is an obvious misprint in the definition of the set T in his proof), who used it to deduce the analogous result about inspheres by means of polarity techniques. Hence we have to study only the case of nonreflexive X.

If X is nonreflexive, by a classical theorem of James [7] there exists $f \in S_1(X^*)$ not attaining its norm. Let $\{y_n\}_1^\infty \subset S_1$ be such that $f(y_n) = 1 - 1/n$, $n = 1, 2, \dots$. Let $e_n = (1 - 1/n)y_n$ and consider the two following subcases:

(i) $\operatorname{diam}\{e_n\}_1^\infty = 2$;

(ii) $\operatorname{diam}\{e_n\}_1^\infty = \delta < 2$.

In case (i), set

$$C = \operatorname{cl}\operatorname{co}(B_{1/2} \cup \{e_n\}_1^\infty).$$

Since $\operatorname{diam} C = 2$ and $C \subset B_1$, S_1 is a circumsphere of C. We claim that $C \cap S_1 = \emptyset$. Suppose, on the contrary, that there exists $x \in C \cap S_1$. Let $x_1, x_2, \dots$ be a sequence converging to x, where for each i it is true that

$$x_i = \lambda_0^i b_i + \sum_{j=1}^{k(i)} \lambda_j^i e_j$$

with

$$b_i \in B_{1/2}, \qquad \lambda_j^i \geqq 0 \text{ for all } j, \qquad \sum_{j=0}^{k(i)} \lambda_j^i = 1.$$

Since $\|x_i\| \to 1$, we must have, for any fixed $j \geqq 0$, $\lambda_j^i \to 0$ when $i \to \infty$. Hence

$$f(x_i) = \lambda_0^i f(b_i) + \sum_{j=1}^{k(i)} \lambda_j^i f(e_j)$$

$$= \lambda_0^i f(b_i) + \sum_{j=1}^{k(i)} \lambda_j^i (1 - 1/j)^2 \to 1 \qquad \text{when } i \to \infty$$

(indeed $\sum_{j=0}^{k(i)} \lambda_j^i = 1$ implies $k(i) \to \infty$ when $i \to \infty$) which is absurd because $f(x) < 1$.

In case (ii), set

$$C = \operatorname{cl}\operatorname{co}(\{-e_n\}_1^\infty \cup B_{1/2} \cup \{e_n\}_1^\infty).$$

Since $\operatorname{diam} C = 2$ and $C \subset B_1$, S_1 is a circumsphere of C. Also in this case we claim that $C \cap S_1 = \emptyset$. For suppose there exists $x \in C \cap S_1$. Let $x_i \to x$, where

$$x_i \in \operatorname{co}(\{-e_n\}_1^\infty \cup B_{1/2} \cup \{e_n\}_1^\infty) = \operatorname{co}(\operatorname{co}\{-e_n\}_1^\infty \cup B_{1/2} \cup \operatorname{co}\{e_n\}_1^\infty),$$

and hence for each i it is true that

$$x_i = \alpha_i a_i + \beta_i b_i + \gamma_i c_i$$

with

$$a_i \in \mathrm{co}\{-e_n\}_1^\infty, \qquad b_i \in B_{1/2}, \qquad c_i \in \mathrm{co}\{e_n\}_1^\infty,$$

$$\alpha_i, \beta_i, \gamma_i \geqq 0, \qquad \alpha_i + \beta_i + \gamma_i = 1.$$

For infinitely many indices i we must have $\alpha_i \leqq \frac{1}{2}$ or $\gamma_i \leqq \frac{1}{2}$. Let, for example, $\alpha_i \leqq \frac{1}{2}$ for infinitely many indices i. Since $\|x_i\| \to 1$, we must have $\beta_i \to 0$. Moreover

$$\inf_i \alpha_i > 0,$$

for if not then

$$\mathrm{dist}(x_i, \mathrm{cl\,co}\{e_n\}_1^\infty) \leqq \inf_i \|x_i - \gamma_i c_i\| \leqq \inf_i (\alpha_i + \beta_i) = 0$$

which would imply $x \in \mathrm{cl\,co}\{e_n\}_1^\infty$ which is absurd because the argument in subcase (i) shows that $\mathrm{cl\,co}\{e_n\}_1^\infty \cap S_1 = \varnothing$ (in fact in subcase (i) we used the fact that $\mathrm{diam}\{e_n\}_1^\infty = 2$ only to prove that S_1 was a circumsphere of C). Now we have, when i goes to infinity,

$$\|x_i\| \leqq \beta_i + \alpha_i \|a_i + c_i\| + |\gamma_i - \alpha_i| \leqq \delta\alpha_i + |1 - 2\alpha_i| + o(1).$$

Then, if $\alpha_i = 1/2$ for infinitely many i, the fact $\delta < 2$ leads immediately to an absurdity, while if $\alpha_i < \frac{1}{2}$ for infinitely many i, then

$$\|x_i\| \leqq 1 + \alpha_i(\delta - 2) + o(1)$$

for such i, which is a contradiction because $\inf_i \alpha_i > 0$. The proof is now complete. ∎

Maybe it would be possible to obtain 3.1 (also for nonreflexive X) from the analogous result on inspheres. In fact, let S_1 be the unique insphere of the centrally symmetric body C and let the transformation $T: X \to X$ map each line through the origin linearly onto itself, and ∂C onto S_1; it's easy to see that $\mathrm{dist}(S_1, T(S_1)) = 0$ but $S_1 \cap T(S_1) = \varnothing$. Nevertheless, $T(S_1)$ may be not convex, and it seems hard to evaluate what happens in dealing with $\mathrm{cl\,co}\, T(S_1)$.

Finally, we mention that, although polarity techniques would naturally seem to be applicable, we have been unable to use them to (a) deduce Theorem 3.1 in the nonreflexive case from Garkavi's characterization [6] of finite-dimensionality in terms of inspheres; (b) pass back and forth between our present characterization of Hilbert spaces in terms of inspheres and the previous characterizations [5] [9] in terms of circumspheres.

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